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Best Approximations to Random Variables Based on Trimming Procedures*

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Let X be a R^n-valued random variable; for a class of suitable nondecreasing functions Φ: $R^+ \to R^+$ and $\alpha \in (0, 1)$, a family of best approximations to X based on trimming procedures is obtained. Existence and a characterization which relates the best approximations and the best trimming sets are obtained. The problem of uniqueness is studied for real valued random variables. © 1991 Academic Press, Inc.

1. INTRODUCTION

Procedures based on trimming a data set and subsequently choosing a best approximation of the remaining set are well known in several branches of mathematics. Perhaps the best known of such procedures is the trimmed mean, of frequent use in statistics and obligatory reference in robust statistics criteria.

However, the problem arises from the arbitrariness which appears in the way one selects the proportion to be trimmed in the left and right sides of the data. On the other hand we take also into account the difficulty in generalizing this procedure to random variables valued in R^n where there do not exist preferential directions for removing data.

This paper deals with obtaining best approximations to random variables based on trimming procedures which both do not depend on arbitrary decisions and can be defined directly for R^n-valued r.v.

The paper will be developed in the general framework of a R^n-valued random variable X defined on a probability space (Ω, σ, P). For a suitable nondecreasing function Φ: $R^+ \to R^+$ and for $\alpha \in (0, 1)$ we look for a Borel

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set B_0 with $P(X \in B_0) \geqslant 1-\alpha$ and a value $m_0 \in R^n$ such that the discrepancy over B_0 between X and m_0, given by

$$\frac{1}{P(X \in B_0)} \int I_{B_0}(X)\, \Phi(\|X-m_0\|)\, dP,$$

becomes as small as possible. To be more precise: We try to obtain B_0 and m_0 satisfying

$$\frac{1}{P_x(B_0)} \int I_{B_0}(X)\, \Phi(\|X-m_0\|)\, dP$$
$$= \inf_{B \in \beta^n, P_x(B) \geqslant 1-\alpha} \inf_{m \in R^n} \frac{1}{P_x(B)} \int I_B(X)\, \Phi(\|X-m\|)\, dP, \qquad (1.1)$$

where $\| \ \|$ denotes the usual norm on R^n, I_A denotes the indicator set function of A, and P_x is the probability measure induced by X on (R^n, β^n). (Similar techniques are employed by Rousseeuw [3] and Rousseeuw and Yohai [4] for trimming data sets in the context of estimation with high breakdown point. Nevertheless the kind of results obtained is very different.)

Note that for fixed $B_0 \in \beta^n$

$$\frac{1}{P_x(B_0)} \int I_{B_0}(X)\, \Phi(\|X-m\|)\, dP = \int \Phi(\|x-m\|)\, dP_x(\cdot/B_0) \qquad (1.2)$$

and then, the solutions of

$$\frac{1}{P_x(B_0)} \int I_{B_0}(X)\, \Phi(\|X-m_0\|)\, dP = \inf_{m \in R^n} \frac{1}{P_x(B_0)} \int I_{B_0}(X)\, \Phi(\|X-m\|)\, dP$$

are the well known Φ-means of $P_x(\cdot/B_0)$. (See Herrndorf [2] and Brøns *et al.* [1] for references of Φ-means, p-means, and other generalized means.)

If B_0 and m_0 are a solution of (1.1) then they will be called a Φ-best trimming set for X at level α and an impartial trimmed Φ-mean of X at level α, respectively.

The word "impartial" means that it is the random variable X itself which provides the best way of trimming it.

The problem can be stated in a more general way by using "trimming" functions instead of trimming sets. A trimming function for X at level α is a β-measurable map τ: $R^n \to [0, 1]$ satisfying $\int \tau(X)\, dP \geqslant 1-\alpha$. The trimming functions explain the degree of participation of each point for "trimming" the r.v. X. When we are working with trimming sets, each point either participates completely or does not participate at all in the construction of the approximation to X. On the other hand when we are handling trimming functions, all possible halfway degrees of participation are available.

Now the problem is to select a $\tau_0 \in T$ and a value $m_0 \in R^n$ such that

$$\frac{1}{\int \tau_0(X)\, dP} \int \tau_0(X)\, \Phi(\|X-m_0\|)\, dP$$
$$= \inf_{\tau \in T} \inf_{m \in R^n} \frac{1}{\int \tau(X)\, dP} \int \tau(X)\, \Phi(\|X-m\|)\, dP, \tag{1.3}$$

where T denotes the class of trimming functions for X at level α.

Obviously $I_B \in T$ for every Borel set B with $P_x(B) \geqslant 1-\alpha$, hence the approximation obtained through trimming functions could be better than the one obtained through trimming sets.

If for fixed $\tau \in T$, P_x^τ denotes the probability on (R^n, β^n) given by

$$P_x^\tau(A) = \frac{\int_A \tau(x)\, dP_x}{\int \tau(x)\, dP_x} \qquad \text{for all} \quad A \in \beta^n,$$

analogously to (1.2) we have

$$\frac{1}{\int \tau_0(X)\, dP} \int \tau_0(X)\, \Phi(\|X-m\|)\, dP = \int \Phi(\|x-m\|)\, dP_x^{\tau_0},$$

so the solutions of

$$\frac{1}{\int \tau_0(X)\, dP} \int \tau_0(X)\, \Phi(\|X-m_0\|)\, dP$$
$$= \inf_{m \in R^n} \frac{1}{\int \tau_0(X)\, dP} \int \tau_0(X)\, \Phi(\|X-m\|)\, dP$$

are the Φ-means of $P_x^{\tau_0}$.

If τ_0 and m_0 are a solution of (1.3) then τ_0 will be called a Φ-best trimming function for X at level α, and m_0, as above, an impartial trimmed Φ-mean of X at level α.

Among the different methods for obtaining best approximations to X based on trimming procedures we also consider those corresponding to L_∞-norms: Choose a set $B_0 \in \beta^n$ with $P(X \in B_0) \geqslant 1-\alpha$ and a value $m_0 \in R^n$ verifying

$$\operatorname*{ess\,sup}_{X \in B_0} \|X-m_0\| = \inf_{B \in \beta^n} \inf_{m \in R^n} \operatorname*{ess\,sup}_{X \in B} \|X-m\|. \tag{1.4}$$

If B_0 and m_0 are a solution of (1.4) they will be called a L_∞-best trimming set for X at level α and an impartial Chebyshev center (CH-center) of X at level α, respectively.

Analogously, if we work with trimming functions we try to obtain τ_0 and m_0 such that

$$\operatorname*{ess\,sup}_{X \in Sop(\tau_0)} \|X - m_0\| = \inf_{\tau \in T} \inf_{m \in R^n} \operatorname*{ess\,sup}_{X \in Sop(\tau)} \|X - m\|, \tag{1.5}$$

where $Sop(\tau) = \{x \in R^n / \tau(x) > 0\}$.

Now, if τ_0 and m_0 are a solution of (1.5) then τ_0 is called a L_∞-best trimming function for X at level α, and m_0 as above an impartial CH-center of X at level α.

The main goal of this paper is to prove that the balls provide the best ways of trimming. In Section 3 we prove the existence of impartial trimmed Φ-means and CH-centers at level α for every random variable X. Moreover, we will prove that the indicator set functions of balls in R^n are essentially the Φ-best trimming functions, and the impartial trimmed Φ-means will be characterized for being the centers of the balls defining the Φ-best trimming functions. In Section 4 we study the real case, and we obtain for r.v. X having a density, that unimodality is a sufficient condition for assuring the uniqueness of the impartial trimmed Φ-means.

2. Notation

In this paper (Ω, σ, P) is a probability space, X is a R^n-valued random variable defined on (Ω, σ, P), β^n is the Borel σ-algebra on R^n, and P_x is the probability measure induced by X on (R^n, β^n). With $\| \ \|$ we will denote the usual norm on R^n, and for $m \in R^n$ and $r \geq 0$, $B(m, r)$ will be the open ball centered at m and with radius r. Moreover, for a set $B \subset R^n$, $\bar{B}$ denotes its closure and B^c its complementary set.

From now on, Φ: $R^+ \to R^+$ will be considered continuous, strictly increasing, and such that $\Phi(0) = 0$.

For $\alpha \in (0, 1)$, $T = T(\alpha, X)$ denotes the set of trimming functions for X at level α, i.e.,

$$T = T(\alpha, X) = \left\{\tau: R^n \to [0, 1] \text{ measurable} \middle/ \int \tau(X)\, dP \geq 1 - \alpha\right\}.$$

Also, for $\beta \leq \alpha$, with T_β we denote the subset of T given by

$$T_\beta = \left\{\tau \in T \middle/ \int \tau(X)\, dP = 1 - \beta\right\}.$$

The minimum values in (1.3) and (1.5) will be denoted by $V_\Phi^\alpha(X)$ and $V_\infty^\alpha(X)$, respectively.

It is obvious that $V_\Phi^\alpha(X) < \infty$ for every Φ, α, and X; in fact, taking a ball $B = B(0, r)$ such that $P_x(B) \geqslant 1 - \alpha$ we have

$$V_\Phi^\alpha \leqslant \frac{1}{P_x(B)} \int I_B(X)\, \Phi(\|X\|)\, dP \leqslant \Phi(r) < \infty,$$

and analogously $V_\infty^\alpha < r < \infty$.

3. Existence of Impartial Trimmed Φ-Means

In this section we will need some additional notation: For fixed $m \in R^n$ and $\beta \leqslant \alpha$, $r_\beta(m)$ will be the radius of the smallest open ball centered at m and verifying $P_x(B(m, r_\beta(m))) \leqslant 1 - \beta \leqslant P_x(\bar{B}(m, r_\beta(m)))$. Moreover, $\tau_{m,\beta}$ will denote a trimming function in T_β verifying

$$I_{B(m,\, r_\beta(m))} \leqslant \tau_{m,\beta} \leqslant I_{\bar{B}(m, r_\beta(m))}.$$

Lemma 1. *Let $m \in R^n$, $\beta \leqslant \alpha$, and $B = B(m, r_\beta(m))$. Then*

(a) $\int \tau_{m,\beta}(X)\, \Phi(\|X - m\|)\, dP \leqslant \int \tau(X)\, \Phi(\|X - m\|)\, dP$ *for all* $\tau \in T_\beta$,

(b) *The equality holds in* (a) *if and only if* $I_B \leqslant \tau \leqslant I_{\bar{B}}$ *a.e.* P_x.

Proof. Take $\tau \in T$ and note that

$$\tau_{m,\beta}(x)(1 - \tau(x)) = 0 \qquad \text{for all} \quad x \notin \bar{B} \tag{3.1}$$

$$\int \tau_{m,\beta}(X)(1 - \tau(X))\, dP = \int \tau(X)(1 - \tau_{m,\beta}(X))\, dP \tag{3.2}$$

$$\tau(x)(1 - \tau_{m,\beta}(x)) = 0 \qquad \text{for all} \quad x \in B. \tag{3.3}$$

Now, applying (3.1) to (3.3) successively we have

$$\int \tau_{m,\beta}(X)(1 - \tau(X))\, \Phi(\|X - m\|)\, dP$$

$$\leqslant \Phi(r_\beta(m)) \int \tau(X)(1 - \tau_{m,\beta}(X))\, dP \tag{3.4}$$

$$= \Phi(r_\beta(m)) \int \tau(X)(1 - \tau_{m,\beta}(x))\, dP$$

$$\leqslant \int \tau(X)(1 - \tau_{m,\beta}(X))\, \Phi(\|X - m\|)\, dP. \tag{3.5}$$

So we have

$$\begin{aligned}\int \tau_{m,\beta}(X)\, \Phi(\|X-m\|)\, dP &= \int \tau_{m,\beta}(X)\, \tau(X)\, \Phi(\|X-m\|)\, dP \\ &\quad + \int \tau_{m,\beta}(X)(1-\tau(X))\, \Phi(\|X-m\|)\, dP \\ &\leqslant \int \tau_{m,\beta}(X)\, \tau(X)\, \Phi(\|X-m\|)\, dP \\ &\quad + \int \tau(X)(1-\tau_{m,\beta}(X))\, \Phi(\|X-m\|)\, dP \\ &= \int \tau(X)\, \Phi(\|X-m\|)\, dP;\end{aligned}$$

moreover, the equality holds if and only if (3.4) and (3.5) are equalities. Now, (3.4) is an equality if and only if

$$\int_B \tau_{m,\beta}(x)(1-\tau(x))\, dP_x = 0$$

i.e.,

$$\int_B (1-\tau(x))\, dP_x = 0,$$

i.e.,

$$I_B \leqslant \tau, \qquad P_x \text{ a.e.}$$

Analogously, (3.5) is an equality if and only if

$$\int_{\bar{B}^c} \tau(x)(1-\tau_{m,\beta}(x))\, dP_x = 0,$$

i.e.,

$$\int_{\bar{B}^c} \tau(x)\, dP_x = 0,$$

i.e.,

$$\tau \leqslant I_{\bar{B}}, \qquad P_x \text{ a.e.}$$

Then, the equality holds in (a) if and only if $I_B \leqslant \tau \leqslant I_{\bar{B}}$ P_x a.e. and the proof is finished. ▮

LEMMA 2. *Let $m \in R^n$ and $\beta \leqslant \alpha$. Then*

(a) $(1/(1-\alpha)) \int \tau_{m,\alpha}(X)\, \Phi(\|X-m\|)\, dP \leqslant (1/(1-\beta)) \int \tau_{m,\beta}(X)\, \Phi(\|X-m\|)\, dP$.

(b) *The equality in* (a) *holds if and only if* $r_\alpha(m) = r_\beta(m)$ *and* $P_x(B(m, r_\alpha(m))) = 0$.

Proof. First note that, for any α in (0, 1), if τ and τ' are in T_α and satisfy

$$I_{B(m, r_\alpha(m))} \leqslant \tau,\ \tau' \leqslant I_{\bar{B}(m, r_\alpha(m))},$$

we have

$$\int \tau(X)\, \Phi(\|X-m\|)\, dP = \int \tau'(X)\, \Phi(\|X-m\|)\, dP.$$

Hence, without loss of generality, we can assume that $\tau_{m,\beta}(X) \geqslant \tau_{m,\alpha}(X)$ P_x-a.s.: In fact, for $\beta \leqslant \alpha$, it is always possible to choose $\tau_{m,\beta}$ and $\tau_{m,\alpha}$ such that $\tau_{m,\beta} \leqslant \tau_{m,\alpha}$ pointwise. Consequently

$$\int (\tau_{m,\beta}(X) - \tau_{m,\alpha}(X))\, \Phi(\|X-m\|)\, dP$$
$$\geqslant \Phi(r_\alpha(m)) \int (\tau_{m,\beta}(X) - \tau_{m,\alpha}(X))\, dP. \tag{3.6}$$

Also we have

$$\Phi(r_\alpha(m)) \int \tau_{m,\alpha}(X)\, dP \geqslant \int \tau_{m,\alpha}(X)\, \Phi(\|X-m\|)\, dP. \tag{3.7}$$

Now, applying (3.6) and (3.7) successively we have

$$\int \tau_{m,\alpha}(X)\, dP \int (\tau_{m,\beta}(X) - \tau_{m,\alpha}(X))\, \Phi(\|X-m\|)\, dP$$
$$\geqslant \int \tau_{m,\alpha}(X)\, dP \Phi(r_\alpha(m)) \int (\tau_{m,\beta}(X) - \tau_{m,\alpha}(X))\, dP$$
$$\geqslant \int \tau_{m,\alpha}(X)\, \Phi(\|X-m\|)\, dP \int (\tau_{m,\beta}(X) - \tau_{m,\alpha}(X))\, dP.$$

So we have

$$\int \tau_{m,\alpha}(X)\,dP \int \tau_{m,\beta}(X)\,\Phi(\|X-m\|)\,dP$$

$$= \int \tau_{m,\alpha}(X)\,dP \int \tau_{m,\alpha}(X)\,\Phi(\|X-m\|)\,dP$$

$$+ \int \tau_{m,\alpha}(X)\,dP \int (\tau_{m,\beta}(X) - \tau_{m,\alpha}(X))\,\Phi(\|X-m\|)\,dP$$

$$\geqslant \int \tau_{m,\alpha}(X)\,dP \int \tau_{m,\alpha}(X)\,\Phi(\|X-m\|)\,dP$$

$$+ \int \tau_{m,\alpha}(X)\,\Phi(\|X-m\|)\,dP \int (\tau_{m,\beta}(X) - \tau_{m,\alpha}(X))\,dP$$

$$= \int \tau_{m,\beta}(X)\,dP \int \tau_{m,\alpha}(X)\,\Phi(\|X-m\|)\,dP,$$

i.e.,

$$(1-\alpha) \int \tau_{m,\beta}(X)\,\Phi(\|X-m\|)\,dP \geqslant (1-\beta) \int \tau_{m,\alpha}(X)\,\Phi(\|X-m\|)\,dP.$$

Moreover, the equality in (a) holds if and only if (3.6) and (3.7) are equalities.

Now, the equality in (3.6) holds if and only if

$$\int_{\bar{B}^c} (\tau_{m,\beta}(x) - \tau_{m,\alpha}(x))\,dP_x = 0$$

(where $B = B(m, r_\alpha(m))$) which holds if and only if $r_\alpha(m) = r_\beta(m)$.

Analogously, (3.7) is an equality if and only if

$$\int_B \tau_{m,\alpha}(x)\,dP_x = 0,$$

i.e.,

$$P_x(B) = 0. \quad \blacksquare$$

PROPOSITION 3. $\inf_{\tau \in T} \inf_{m \in R^n} (1/\int \tau(X)\,dP) \int \tau(X)\,\Phi(\|X-m\|)\,dP = \inf_{m \in R^n} (1/(1-\alpha)) \int \tau_{m,\alpha}(X)\,\Phi(\|X-m\|)\,dP$.

Proof. Let $\tau \in T$ and $m \in R^n$. Applying successively Lemma 2 and Lemma 1 we obtain

$$\frac{1}{1-\alpha}\int \tau_{m,\alpha}(X)\,\Phi(\|X-m\|)\,dP \leqslant \frac{1}{\int \tau(X)\,dP}\int \tau(X)\,\Phi(\|X-m\|)\,dP$$

hence Proposition 3 holds. ∎

The above result will be very useful for proving the existence of impartial trimmed Φ-means.

THEOREM 4 (Existence). *Let X be a r.v. defined on (Ω, σ, P) and valued in R^n. Let $\alpha \in (0, 1)$ and let Φ: $R^+ \to R^+$ a continuous strictly increasing function such that $\Phi(0) = 0$. Then there exists an impartial trimmed Φ-mean of X at level α.*

Proof. From Proposition 3 we can take a sequence $\{m_n\} \subset R^n$ such that

$$\frac{1}{1-\alpha}\int \tau_{m_n,\alpha}(X)\,\Phi(\|X-m_n\|)\,dP \downarrow V_\Phi^\alpha. \tag{3.8}$$

To simplify the notation we will denote, for every $n \in N$,

$$\tau_n = \tau_{m_n,\alpha}, \qquad r_n = r_\alpha(m_n), \qquad \text{and} \qquad B_n = B(m_n, r_n).$$

It is easy to see that $\{m_n\}$ and $\{r_n\}$ are bounded sequences and, therefore, we can obtain convergent subsequences which we denote as the initial ones.

Hence we have

$$m_n \xrightarrow[n\to\infty]{} m_0 \in R^n \qquad \text{and} \qquad r_n \xrightarrow[n\to\infty]{} r_0 \in R^+.$$

Let us denote $B_0 = B(m_0, r_0)$, i.e., the limit ball. Note that

$$I_{B_0}(X) \leqslant \underline{\lim}\ \tau_n(X) \leqslant \overline{\lim}\ \tau_n(X) \leqslant I_{\bar{B}_0}$$

hence Fatou's Lemma implies that

$$\int I_{B_0}(X)\,dP \leqslant \int \underline{\lim}\ \tau_n(X)\,dP \leqslant 1-\alpha \leqslant \int \overline{\lim}\ \tau_n(X)\,dP \leqslant \int I_{\bar{B}_0}(X)\,dP,$$

i.e.,

$$r_0 = r_\alpha(m_0) \qquad \text{and} \qquad B_0 = B(m_0, r_0).$$

Let us denote $\tau_0 = \tau_{m_0,\alpha}$. We will prove that τ_0 and m_0 satisfy

$$\left| \int \tau_n(X)\, \Phi(\|X - m_n\|)\, dP - \int \tau_0(X)\, \Phi(\|X - m_0\|)\, dP \right| \xrightarrow[n \to \infty]{} 0 \quad (3.9)$$

which implies that

$$\frac{1}{1-\alpha} \int \tau_0(X)\, \Phi(\|X - m_0\|)\, dP = V_\Phi^\alpha.$$

It is obvious that

$$\begin{aligned}
&\left| \int \tau_n(X)\, \Phi(\|X - m_n\|)\, dP - \int \tau_0(X)\, \Phi(\|X - m_0\|)\, dP \right| \\
&\quad = \left| \int (\tau_n(X)\, \Phi(\|X - m_n\|) - \tau_0(X)\, \Phi(\|X - m_0\|))\, dP \right| \\
&\quad = \left| \int \tau_n(X)(\Phi(\|X - m_n\|) - \Phi(\|X - m_0\|))\, dP \right. \\
&\qquad \left. + \int (\tau_n(X) - \tau_0(X))\, \Phi(\|X - m_0\|)\, dP \right| \\
&\quad \leqslant \left| \int \tau_n(X)(\Phi(\|X - m_n\|) - \Phi(\|X - m_0\|))\, dP \right| \\
&\qquad + \left| \int (\tau_n(X) - \tau_0(X))\, \Phi(\|X - m_0\|)\, dP \right| \\
&\quad = A_n + K_n,
\end{aligned}$$

hence it suffices to prove that the sequences $\{A_n\}$ and $\{K_n\}$ converge to 0.

Since Φ is uniformly continuous on the compact set $[0, \sup_n r_n]$, we have

$$\begin{aligned}
A_n &= \left| \int \tau_n(X)(\Phi(\|X - m_n\|) - \Phi(\|X - m_0\|))\, dP \right| \\
&\leqslant \int \tau_n(X)\, |\Phi(\|X - m_n\|) - \Phi(\|X - m_0\|)|\, dP \\
&\leqslant (1-\alpha) \sup_{X \in \bar{B}_n} |\Phi(\|X - m_n\|) - \Phi(\|X - m_0\|)| \xrightarrow[n \to \infty]{} 0.
\end{aligned}$$

Let us denote $C_n = \{x/\tau_n(x) > \tau_0(x)\}$ and $D_n = \{x/\tau_n(x) < \tau_0(x)\}$. We have

$$0=\int (\tau_n(x)-\tau_0(x))\,dP_x$$

$$=\int_{C_n} (\tau_n(x)-\tau_0(x))\,dP_x+\int_{D_n} (\tau_n(x)-\tau_0(x))\,dP_x$$

and then

$$\int_{C_n} (\tau_n(x)-\tau_0(x))\,dP_x=\int_{D_n} (\tau_0(x)-\tau_n(x))\,dP_x. \tag{3.10}$$

Now, taking into account that $C_n \subset B_0^c \cap \bar{B}_n$ and $D_n \subset \bar{B}_0 \cap B_n^c$ for every n, we have

$$\Phi(r_0) \leqslant \Phi(\|x-m_0\|) \leqslant \Phi(r_n+\|m_n-m_0\|) \qquad \text{for all} \quad x \in C_n \tag{3.11}$$

and

$$\Phi(r_0) \geqslant \Phi(\|x-m_0\|) \geqslant \Phi(r_n-\|m_n-m_0\|) \qquad \text{for all} \quad x \in D_n. \tag{3.12}$$

So applying (3.10) to (3.12) we obtain

$$\int (\tau_n(X)-\tau_0(X))\,\Phi(\|X-m_0\|)\,dP$$

$$=\int_{C_n} (\tau_n(x)-\tau_0(x))\,\Phi(\|x-m_0\|)\,dP_x$$

$$-\int_{D_n} (\tau_0(x)-\tau_n(x))\,\Phi(\|x-m_0\|)\,dP_x$$

$$\geqslant \Phi(r_0)\int_{C_n} (\tau_n(x)-\tau_0(x))\,dP_x$$

$$-\Phi(r_0)\int_{D_n} (\tau_0(x)-\tau_n(x))\,dP_x=0,$$

hence applying once more (3.10) to (3.12) we obtain

$$K_n=\left|\int (\tau_n(X)-\tau_0(X))\,\Phi(\|X-m_0\|)\,dP\right|$$

$$=\int (\tau_n(X)-\tau_0(X))\,\Phi(\|X-m_0\|)\,dP$$

$$= \int_{C_n} (\tau_n(x) - \tau_0(x))\, \Phi(\|x - m_0\|)\, dP_x$$

$$- \int_{D_n} (\tau_0(x) - \tau_n(x))\, \Phi(\|x - m_0\|)\, dP_x$$

$$\leqslant \Phi(r_n + \|m_n - m_0\|) \int_{C_n} (\tau_n(x) - \tau_0(x))\, dP_x$$

$$- \Phi(r_n - \|m_n - m_0\|) \int_{D_n} (\tau_0(x) - \tau_n(x))\, dP_x$$

$$\leqslant \Phi(r_n + \|m_n - m_0\|) - \Phi(r_n - \|m_n - m_0\|) \xrightarrow[n \to \infty]{} 0,$$

and the proof is complete. ∎

As a consequence of Lemma 1 we obtain a very important relationship between the impartial trimmed Φ-means and the Φ-best trimming functions:

THEOREM 5. *Under the hypotheses of Theorem* 4, *if* τ_0 *and* m_0 *are a solution of* (1.3) *and* $B = B(m_0, r_\alpha(m_0))$ *then*

$$I_B \leqslant \tau_0 \leqslant I_{\bar{B}}, \qquad P_x \text{ a.e.} \tag{3.13}$$

Proof. On the contrary, suppose that (3.13) is not true. Then Lemma 1(b) implies that

$$\int \tau_{m_0, \alpha}(X)\, \Phi(\|X - m_0\|)\, dP < \int \tau_0(X)\, \Phi(\|X - m_0\|)\, dP$$

so (τ_0, m_0) cannot be a solution of (1.3). ∎

COROLLARY 6. *Under the hypotheses of Theorem* 5, *if* P_x *is absolutely continuous with respect to the Lebesgue measure on* R^n, *then*

$$I_B = \tau_0, \qquad P_x \text{ a.e.}$$

Definitively we have proved that the balls provide the best ways of trimming random variables. Therefore, roughly speaking, the search of the best trimming sets and functions can be restricted to the balls and the indicator set functions of the balls, respectively. Moreover we know that there exists a double relationship between the impartial trimmed Φ-means and the Φ-best trimming functions: The impartial trimmed Φ-means of X at level α are the centers of the balls defining the Φ-best trimming functions for X at level α and also the Φ-means of P_x "restricted" to such balls.

All the results in this section are true in the case of L_∞-approximation and the proofs are obvious. In fact these results become:

LEMMA 1′. *Let $m \in R^n$, $\beta \leqslant \alpha$, and $B = B(m, r_\beta(m))$. Then*

(a) $\operatorname{ess\,sup}_{X \in Sop(\tau_{m,\beta})} \|X - m\| \leqslant \operatorname{ess\,sup}_{X \in Sop(\tau)} \|X - m\|$ *for all* $\tau \in T_\beta$.

(b) *The equality holds in* (a) *if and only if* $\tau \leqslant I_{\bar{B}}$ *a.e.* P_x.

LEMMA 2′. *Let $m \in R^n$ and $\beta \leqslant \alpha$. Then*

(a) $\operatorname{ess\,sup}_{X \in Sop(\tau_{m,\alpha})} \|X - m\| \leqslant \operatorname{ess\,sup}_{X \in Sop(\tau_{m,\beta})} \|X - m\|$.

(b) *The equality in* (a) *holds if and only if* $r_\alpha(m) = r_\beta(m)$.

PROPOSITION 3′. $\inf_{\tau \in T} \inf_{m \in R^n} \operatorname{ess\,sup}_{X \in Sop(\tau)} \|X - m\| = \inf_{m \in R^n} r_\alpha(m)$.

THEOREM 4′ (Existence). *Let X be a r.v. defined on (Ω, σ, P) and valued in R^n and let $\alpha \in (0, 1)$. Then there exists an impartial trimmed CH-center of X at level α.*

THEOREM 5′. *Under the hypotheses of Theorem* 4′, *if τ_0 and m_0 are solutions of* (1.5) *and $B = B(m_0, r_\alpha(m_0))$ then*

$$\tau_0 \leqslant I_{\bar{B}}, \qquad P_x \text{ a.e.}$$

COROLLARY 6′. *In the hypotheses of Theorem* 5′, *if P_x is absolutely continuous with respect to the Lebesgue measure on R^n, then*

$$I_B = \tau_0, \qquad P_x \text{ a.e.}$$

Remark 7. The advantages of working with trimming functions instead of trimming sets are very important:

(i) Obviously, sometimes there do not exist Borel sets with P_x-measure exactly $1 - \alpha$.

(ii) There exist random variables whose Φ-best trimming sets are neither a ball nor a convex set.

EXAMPLE. Let X be a real valued random variable such that $P[X = 0] = 1/2$, $P[X = 1] = 3/8$, and $P[X = 6/5] = 1/8$. Let $\alpha = 3/8$ and let $\Phi(t) = t^2$. It is easy to see that the Φ-best trimming sets for X at level α have to contain $\{0, 6/5\}$ and 1 cannot belong to them.

(iii) The Φ-best trimming set has no universal bound.

EXAMPLE. Let X be a real valued random variable with the probability law given by

$$P[X=a]=e^{-1}$$
$$P[X=k]=e^{-1}/k!, \qquad k=1, 2, ...$$

For $\alpha=e^{-1}$ and $\Phi(t)=t^2$ we can see that for $a\to-\infty$ the Φ-best trimming sets for X at level α have to contain $\{1, 2, ..., \infty\}$.

Remark 8. For random variables with the probability law absolutely continuous with respect to the Lebesgue measure on (R^n, β^n), trimming functions and trimming sets provide the same results.

Remark 9. If we consider the case $\alpha=0$ then the best approximations obtained are the Φ-means of X. The case $\alpha=1$ has no sense; however, we can study what does happen for $\alpha\to 1$. Let $\{\alpha_n\}\subset(0, 1)$ such that $\alpha_n\to 1$. For every $n\geqslant 1$, let m_n be an impartial trimmed Φ-mean (resp. CH-center) of X at level α. Suppose that m is an accumulation point of $\{m_n, n\geqslant 1\}$. It is easy to see that:

(i) If $S=\{x\in R^n/P[X=x]>0\}\neq\varnothing$ then $m\in S$.

(ii) If P_x has a density f then $f(m)\geqslant f(x)$ for all $x\in R^n$.

4. UNIQUENESS IN THE CASE OF REAL VALUED RANDOM VARIABLES

Throughout this section X is a real valued r.v. defined on (Ω, σ, P) with distribution function F. Moreover we will suppose some additional conditions for the function Φ:

$$\Phi \text{ has a derivative } \Psi(t)=\frac{d}{dt}\Phi(t) \tag{4.1}$$

$$\Psi \text{ has a derivative } \Psi'(t)=\frac{d}{dt}\Psi(t). \tag{4.2}$$

Since Φ is strictly increasing, hence $\Psi(t)>0$ for every $t\in R^+$.

For every $m\in R$, $(l(m), u(m))$ will denote the shortest interval centered at m $(m=(l(m)+u(m))/2)$ and verifying

$$P_x((l(m), u(m)))\leqslant 1-\alpha\leqslant P_x([l(m), u(m)]).$$

Because of Theorem 5, if m is an impartial trimmed Φ-mean of X at level

α then the interval $(l(m), u(m))$ defines the Φ-best trimming functions associated to m; i.e., if (τ, m) is a solution of (1.3) then

$$I_{(l(m), u(m))} \leqslant \tau \leqslant I_{[l(m), u(m)]}, \qquad P_x \text{ a.e.} \tag{4.3}$$

We will prove that such Φ-best trimming functions are essentially equal; i.e., for m being an impartial trimmed Φ-mean of X at level α, there exists essentially a unique Φ-best trimming function such that

$$\frac{1}{1-\alpha} \int \tau(X)\, \Phi(|X-m|)\, dP = V_\Phi^\alpha.$$

PROPOSITION 10. *Let m be an impartial trimmed Φ-mean of X at level α. If τ_1 and τ_2 are Φ-best trimming functions for X at level α satisfying* (4.3) *then*

$$\tau_1 = \tau_2, \qquad P_x \text{ a.e.}$$

Proof. Since

$$\int \tau_i(x)\, \Phi(|x-m|)\, dP_x = \min_m \int \tau_i(x)\, \Phi(|x-m|)\, dP_x, \qquad i=1,2$$

hence

$$\int \tau_i(x)\, \Psi(|x-m|) \operatorname{sign}(x-m)\, dP_x = 0, \qquad i=1,2$$

and then

$$\int (\tau_1(x) - \tau_2(x))\, \Psi(|x-m|) \operatorname{sign}(x-m)\, dP_x = 0,$$

i.e.,

$$(\tau_1(l(m)) - \tau_2(l(m)))\, P[X = l(m)] = (\tau_1(u(m)) - \tau_2(u(m)))\, P[X = u(m)].$$

Hence at least one of the following is true:

(a) $\tau_1(l(m)) = \tau_2(l(m))$ and $\tau_1(u(m)) = \tau_2(u(m))$

(b) $\tau_1(l(m)) = \tau_2(l(m))$ and $P[X = u(m)] = 0$

(c) $\tau_1(u(m)) = \tau_2(u(m))$ and $P[X = l(m)] = 0$

and then

$$\tau_1 = \tau_2, \qquad P_x \text{ a.e.} \quad \blacksquare$$

As a consequence of the above proposition we have the following result about the interval defining a Φ-best trimming function for X at level α:

COROLLARY 11. *Let m be an impartial trimmed Φ-mean of X at level α. Then at least one of the following is true:*

(a) $P[X \in (l(m), u(m))] = 1 - \alpha$

(b) $P[X \in [l(m), u(m)]] = 1 - \alpha$

(c) $P[X = l(m)] = 0$

(d) $P[X = u(m)] = 0.$

Proof. In fact, if (a)–(d) are false then there exist trimming functions τ_1 and τ_2 such that $P_x[\tau_1 \neq \tau_2] > 0$ and

$$I_{(l(m), u(m))} \leqslant \tau_1, \tau_2 \leqslant I_{[l(m), u(m)]} \qquad P_x \text{ a.e.}$$

which contradicts Proposition 10. ∎

Remark 12. The above results are not true for impartial trimmed CH-centers. Counterexamples with discrete random variables are obvious.

Remark 13. We conjecture that analogous results are true for R^n-valued random variables; i.e., for each impartial trimmed Φ-mean of X at level α there exists essentially a unique Φ-best trimming function and, therefore, the ball B defining such a Φ-best trimming function satisfies one of the following:

(a) $P[X \in B] = 1 - \alpha.$

(b) $P[X \in \bar{B}] = 1 - \alpha.$

(c) There exists $x_0 \in Bd(B)$ such that $P[X \in Bd(B)] = P[X = x_0]$, where $Bd(B)$ denotes the boundary of B.

Now the goal is to prove that for real valued random variables X having a density function f, unimodality is a sufficient condition for the uniqueness of the impartial trimmed Φ-means and CH-centers.

THEOREM 14 (Uniqueness). *Let X be a real valued r.v. X defined on (Ω, σ, P), with distribution function F having a differentiable density f which is unimodal and satisfies $f(x) > 0$ for all $x \in R$. Let $\alpha \in (0, 1)$. Then:*

(a) *For every convex and strictly increasing function Φ with $\Phi(0) = 0$ and satisfying* (4.1) *and* (4.2), *there exists a unique impartial trimmed Φ-mean of X at level α.*

(b) *There exists a unique impartial trimmed CH-center of X at level α.*

Proof. From Proposition 3 the impartial trimmed Φ-means of X at level α are the solutions of

$$\int_{2m - u(m)}^{u(m)} \Phi(|t - m|)\, f(t)\, dt = V_\Phi^\alpha,$$

with $u(m)$ the solution of

$$F(u(m)) - F(2m - u(m)) = 1 - \alpha. \tag{4.4}$$

Let us denote

$$V(m) = \int_{2m-u(m)}^{u(m)} \Phi(|t-m|)\, f(t)\, dt; \tag{4.5}$$

we will prove that there is a unique solution of $V'(m) = 0$.

By deriving in (4.5) we obtain

$$\begin{aligned} V'(m) &= \Phi(|u(m)-m|)\, f(u(m))\, u'(m) \\ &\quad - \Phi(|m-u(m)|)\, f(2m-u(m))(2-u'(m)) \\ &\quad - \int_{2m-u(m)}^{u(m)} \Psi(|t-m|)\, \mathrm{sign}(t-m)\, f(t)\, dt. \end{aligned}$$

Moreover, by deriving in (4.5) we have

$$u'(m)\, f(u(m)) = (2-u'(m))\, f(2m-u(m)) \tag{4.6}$$

and then

$$V'(m) = -\int_{2m-u(m)}^{u(m)} \Psi(|t-m|)\, \mathrm{sign}(t-m)\, f(t)\, dt. \tag{4.7}$$

Let M be the mode of f. Since $f(M) > f(u(M))$ and $f(M) > f(2M-u(M))$ hence $f(m) > f(u(m))$ for $m \in (M_1, M)$ and $f(m) > f(2m-u(m))$ for $m \in (M, M_2)$ being

$$M_1 = \inf\{m \leqslant M / f(m) > f(u(m))\}$$

and

$$M_2 = \sup\{m \geqslant M / f(m) > f(2m-u(m))\}.$$

In the following we outline the proof in four steps.

Step 1. $V'(m) < 0$ for $m < M_1$.

In fact, applying a trivial change of variable we obtain

$$\begin{aligned} V'(m) &= -\int_{2m-u(m)}^{u(m)} \Psi(|t-m|)\, \mathrm{sign}(t-m)\, f(t)\, dt \\ &= \int_{2m-u(m)}^{m} \Psi(m-t)\, f(t)\, dt - \int_{m}^{u(m)} \Psi(t-m)\ \ f(t)\, dt \\ &= -\int_{m}^{u(m)} \Psi(t-m)(f(t) - f(2m-t))\, dt < 0. \end{aligned}$$

Step 2. $V'(m)$ is strictly increasing for $m \in [M_1, M]$.
In fact, by deriving in (4.7) and applying again (4.6) we obtain

$$\begin{aligned}V''(m) &= -\Psi(u(m)-m)\frac{4f(u(m))\,f(2m-u(m))}{f(u(m))+f(2m-u(m))}\\ &\quad+\int_{2m-u(m)}^{u(m)} \Psi'(|t-m|)\,f(t)\,dt\\ &= \Psi(u(m)-m)\frac{(f(u(m)-f(2m-u(m)))^2}{f(u(m))+f(2m-u(m))}\\ &\quad-\int_{2m-u(m)}^{u(m)} \Psi(|t-m|)\,\operatorname{sign}(t-m)\,f'(t)\,dt\\ &= \Psi(u(m)-m)\frac{(f(u(m))-f(2m-u(m)))^2}{f(u(m))+f(2m-u(m))}\\ &\quad+\int_{2m-u(m)}^{m} \Psi(m-t)\,f'(t)\,dt-\int_{m}^{u(m)} \Psi(t-m)\,f'(t)\,dt.\end{aligned}$$

Note that the unimodality of f implies that $f'(x) > 0$ for every $x < M$ and $f'(x) < 0$ for every $x > M$. Then, it suffices to prove that the last integral is negative.

Finally, applying once more the unimodality of f taking into account that Ψ is increasing we obtain the inequalities

$$\int_m^M \Psi(t-m)\,f'(t)\,dt \leqslant \Psi(M-m)(f(M)-f(m))$$

and

$$\begin{aligned}\int_M^{u(m)} \Psi(t-m)\,f'(t)\,dt &\leqslant \Psi(M-m)(f(u(m))-f(M))\\ &< \Psi(M-m)(f(m)-f(M))\end{aligned}$$

which imply

$$\int_m^{u(m)} \Psi(t-m)\,f'(t)\,dt = \int_m^M \Psi(t-m)\,f'(t)\,dt + \int_M^{u(m)} \Psi(t-m)\,f'(t)\,dt < 0.$$

With similar techniques we can see that:

Step 3. $V'(m)$ is strictly increasing on $[M, M_2]$.

Step 4. $V'(m) > 0$ for $m > M_2$.

Then there exists a unique solution of $V'(m)=0$ and the proof of (a) is finished.

For proving (b) note that the impartial trimmed CH-centers of X at level α are the solutions of

$$u(m)-m=V_{\infty}^{\alpha}$$

with $u(m)$ the solution of (4.4).

Let us denote $V_{\infty}(m)=u(m)-m$. From (4.6) we have

$$V'_{\infty}(m)=u'(m)-1=\frac{f(2m-u(m))-f(u(m))}{f(2m-u(m))+f(u(m))}$$

and the unimodality of f assures the uniqueness of the solution of $V'_{\infty}(m)=0$. Moreover, such a solution is characterized by verifying

$$f(u(m))=f(2m-u(m)). \quad \blacksquare$$

Remark 15. If the condition "$f(x)>0$ for all $x \in R$" is removed, then Theorem 14 is also true. In the proof, some caution with the points where $V'(m)$ is not defined is necessary.

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